

Guide to Integration

Mathematics 101

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- 1 Elementary Integrals
- 2 Substitution
- 3 Trigonometric integrals
- 4 Integration by parts
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Table of integrals

Recognise these from a table of derivatives.

The very basics

$$① \int 1 \, dx = x + c$$

$$② \int \frac{1}{x} \, dx = \log|x| + c$$

$$③ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$

$$④ \int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

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$$\textcircled{1} \int 1 \, dx = x + c$$

$$\textcircled{2} \int \frac{1}{x} \, dx = \log|x| + c \text{ — don't forget the } | \cdot |.$$

$$\textcircled{3} \int x^n \, dx = \frac{1}{n+1} x^{n+1} + c \text{ — if } n \neq -1.$$

$$\textcircled{4} \int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

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$$\textcircled{1} \int \sin(ax) \, dx = \frac{-1}{a} \cos(ax) + c$$

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$$\textcircled{3} \int \sec^2(ax) \, dx = \frac{1}{a} \tan(ax) + c$$

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Inverse trig

$$\textcircled{1} \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}(x/a) + c$$

$$\textcircled{2} \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}(x/a) + c.$$

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Substitution rule

From the chain rule we get

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Look for a function and its derivative in the integrand.

Substitution example

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WARNING — you must turn **all** the x 's into the new variable.

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Of course the answers are the same.

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Useful trig identities

$$\cos^2 x + \sin^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

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- If a or b is odd then use

$$\cos^2 x = 1 - \sin^2 x$$

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Now set $u = \sec x + \tan x$:

$$\frac{du}{dx} = \sec x \tan x + \sec^2 x = \sec x (\tan x + \sec x)$$

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Hence we have

$$\int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{1}{u} \cdot \frac{du}{dx} dx$$

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From the product rule we get

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- Frequently used when you have the product of 2 different types of functions.
- You have to choose $f(x)$ and $g'(x)$ — there are 2 options.
- Usually one will work and the other will not.

Integration by parts

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$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx$$

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x \cdot 1 dx \\ &= xe^x - e^x + c\end{aligned}$$

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- This is not getting easier, so stop!

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$$\begin{aligned}\int f(x)g'(x) \, dx &= f(x)g(x) - \int g(x)f'(x) \, dx \\ \int \log x \, dx &= x \log x - \int x/x \, dx \\ &= x \log x - x + c\end{aligned}$$

Trigonometric substitutions

Based on

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

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Things to associate

If the integrand contains

$$\sqrt{a^2 - x^2} \longrightarrow \sin^2 \theta = 1 - \cos^2 \theta$$

$$a^2 + x^2 \longrightarrow 1 + \tan^2 \theta = \sec^2 \theta$$

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- We aren't done yet — we have to change back to the x variable.

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- We can express $\tan \theta$ in terms of $\sin \theta$

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} \\ &= \frac{x/\sqrt{5}}{\sqrt{1 - x^2/5}} = \frac{x}{\sqrt{5}\sqrt{1 - x^2/5}} = \frac{x}{\sqrt{5 - x^2}}\end{aligned}$$

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- Hence the integral is

$$\int (5 - x^2)^{-3/2} dx = \frac{x}{5\sqrt{5 - x^2}} + c$$

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- We have assumed $\sec \theta > 0$. We did similarly in the previous example.

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- So now we need to rewrite in terms of x .
- The $\tan \theta = x/2$ is easy. But $\sec \theta$ is harder:

$$\sec^2 \theta = 1 + \tan^2 \theta$$

$$\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + x^2/4}$$

Trigonometric substitutions

Compute $\int \frac{1}{\sqrt{4+x^2}} dx$

- We substituted $x = 2 \tan \theta$ and got

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}} dx &= \int \sec \theta d\theta \\ &= \log |\sec \theta + \tan \theta| + c \quad \text{previous work}\end{aligned}$$

- So now we need to rewrite in terms of x .
- The $\tan \theta = x/2$ is easy. But $\sec \theta$ is harder:

$$\begin{aligned}\sec^2 \theta &= 1 + \tan^2 \theta \\ \sec \theta &= \sqrt{1 + \tan^2 \theta} = \sqrt{1 + x^2/4}\end{aligned}$$

- Hence: $\int \frac{1}{\sqrt{4+x^2}} dx = \log |\sqrt{1+x^2/4} + x/2| + c.$

Trigonometric substitutions

- Sometimes you need to complete the square in order to get started.

Try $\frac{dx}{4x^2 + 12x + 13}$

Partial fractions

Based on partial fraction decomposition of rational functions

- There are some very general rules for this technique.
- It is one of the few very formulaic techniques of integration.

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-
- Any polynomial with real coefficients can be factored into linear and quadratic factors with real coefficients

$$Q(x) = k(x - a_1)^{m_1}(x - a_2)^{m_2} \cdots (x - a_j)^{m_j} \\ \times (x^2 + b_1x + c_1)^{n_1}(x^2 + b_2x + c_2)^{n_2} \cdots (x^2 + b_lx + c_l)^{n_l}$$

Partial fractions

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$$f(x) = \frac{A_{11}}{(x - a_1)^1} + \frac{A_{12}}{(x - a_1)^2} + \dots + \frac{A_{1m_1}}{(x - a_1)^{m_1}}$$

+ similar terms for each linear factor

$$+ \frac{B_{11}x + C_{11}}{(x^2 + b_1x + c_1)^1} + \frac{B_{12}x + C_{12}}{(x^2 + b_1x + c_1)^2} + \dots + \frac{B_{1n_1}x + C_{1n_1}}{(x^2 + b_1x + c_1)^{n_1}}$$

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- Once in this form, we can integrate term-by-term.

Partial fractions

$$\int \frac{dx}{x(x-1)}$$

- Write in partial fraction form

$$\begin{aligned}\frac{1}{x(x-1)} &= \frac{A}{x} + \frac{B}{x-1} \\ &= \frac{A(x-1) + Bx}{x(x-1)}\end{aligned}$$

Now find A and B .

Compare numerators

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- Hence we have 2 equations

$$\left. \begin{array}{l} A+B = 0 \\ -A = 1 \end{array} \right\} \Rightarrow A = -1, B = 1$$

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always check this!

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Try $\int \frac{1}{x^2 - a^2} dx$.

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- Integrate term-by-term

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$$\begin{aligned}\int \frac{1}{x(x-1)^2} dx &= \int \frac{1}{x} dx + \int \frac{-1}{x-1} dx + \int \frac{1}{(x-1)^2} dx \\ &= \log|x| - \log|x-1| - \frac{1}{x-1} + c \\ &= \log \left| \frac{x}{x-1} \right| + c\end{aligned}$$

About 100 Integrals ...

25. $\int \cos^2 x \sin^2 x dx$

27. $\int \ln |x+2e^x| dx$

29. $\int \cos^2 x \sin^2 x dx$

31. $\int_{-\infty}^{\infty} \cos^2 x \sin^2 x dx$

33. $\int \cos^2 x \sin^2 x dx$

35. $\int \cos^2 x dx$

37. $\int \cos^2 x dx$

33. $\int \frac{1}{(x^2+1)^2} dx$

35. $\int \frac{1}{x^2+1} dx$

37. $\int \frac{1}{x^2+1} dx$

26. $\int_0^{\infty} \cos^2 x dx$

28. $\int \cos^2 x dx$

30. $\int \cos^2 x dx$

32. $\int \cos^2 x dx$

34. $\int \cos^2 x dx$

36. $\int \cos^2 x dx$

38. $\int \cos^2 x dx$

34. $\int \frac{1}{x^2+1} dx$

36. $\int \frac{1}{x^2+1} dx$

38. $\int \frac{1}{x^2+1} dx$

1. $\int \frac{1}{x+1} dx$

5. $\int \frac{1}{x^2+1} dx$

3. $\int \frac{1}{x^2+1} dx$

7. $\int \frac{1}{x^2+1} dx$

9. $\int \frac{1}{x^2+1} dx$

15. $\int_0^{\infty} \frac{1}{\sqrt{x^2+1}} dx$

17. $\int \frac{1}{\sqrt{x^2+1}} dx$

19. $\int \frac{1}{\sqrt{x^2+1}} dx$

21. $\int \frac{1}{\sqrt{x^2+1}} dx$

23. $\int \frac{1}{\sqrt{x^2+1}} dx$

25. $\int \frac{1}{\sqrt{x^2+1}} dx$

27. $\int \frac{1}{\sqrt{x^2+1}} dx$

29. $\int \frac{1}{\sqrt{x^2+1}} dx$

31. $\int \frac{1}{\sqrt{x^2+1}} dx$

39. $\int \frac{\ln x}{x^2} dx$

41. $\int \cos^2 x dx$

43. $\int \cos^2 x dx$

45. $\int \cos^2 x dx$

47. $\int \cos^2 x dx$

49. $\int \cos^2 x dx$

51. $\int \cos^2 x dx$

43. $\int \frac{1}{x^2+1} dx$

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9. $\int \sqrt{x} dx$

11. $\int \sqrt{x} dx$

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17. $\int \sqrt{x} dx$

19. $\int \sqrt{x} dx$

21. $\int \sqrt{x} dx$

2. $\int x \cos^2 x dx$

4. $\int x \cos^2 x dx$

6. $\int x \cos^2 x dx$

8. $\int x \cos^2 x dx$

10. $\int x \cos^2 x dx$

12. $\int x \cos^2 x dx$

14. $\int x \cos^2 x dx$

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